

MATH 1104

FALL 2007

LECTURE SLIDES

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(These Slides replace neither the Text Book nor the Lectures)

Part 1

- Systems of Linear Equations.
- Elementary Row Operations.
- Row Echelon Forms.
- Vectors Equations.
- The Matrix Equations $Ax = b$.
- Solution Sets of Linear Systems.
- Linear Dependence and Independence.

SYSTEMS OF LINEAR EQUATIONS

A system of linear equations (or a linear System) is a collection of one or more linear equations involving the same set of variables, say x_1, x_2, \dots, x_n .

Example:

$$\begin{aligned}2x_1 - x_2 - 3x_3 &= -1 \\ -2x_1 + 2x_2 + 5x_3 &= 3.\end{aligned}$$

is a linear system and $(x_1, x_2, x_3) = (1, 0, 1)$ is a solution for this system.

$(x_1, x_2, x_3) = (2, -4, 3)$ is also a solution for this system.

The set of all possible solutions is called the **solution set** of the linear system.

Two linear systems are called **equivalent** if they have the same solution set.

Example:

$$\begin{array}{l|l}x + y = 4 & 8x + 2y = 26 \\ x - y = 2 & 13x + 3y = 42\end{array}$$

are equivalent systems since the solution set for both of them is $\{(3, 1)\}$.

Which of the following system(s) is/are linear?

$$\begin{aligned}x^2 + y &= 7 \\ x - 5y &= 10\end{aligned}$$

$$\begin{aligned}3x + xy &= 6 \\ x - y &= 3\end{aligned}$$

$$\begin{aligned}\sqrt{8}x + y &= 4 \\ 2x - 3y &= 5\end{aligned}$$

Geometrically, solution of two linear equations in two variables is the intersection of two lines.

Example: Give a geometric representation of the following system of equations.

$$\begin{aligned}l_1 : \quad & x + y = 3 \\ l_2 : \quad & x - y = 1\end{aligned}$$

Solution: $(2, 1)$ is the intersection point, and it is the only solution.

In this case, there is a unique solution.

Example: Give a geometric representation of the following system of equations.

$$\begin{aligned}l_1 : \quad & -x + 2y = 1 \\ l_2 : \quad & x - 2y = -3\end{aligned}$$

Solution: Lines are parallel, no intersection. That is, there is no solution of the given system.

Example: Give a geometric representation of the following system of equations.

$$\begin{aligned} l_1 : \quad 2x - y &= 1 \\ l_2 : \quad -4x + 2y &= -2 \end{aligned}$$

Solution: l_1 and l_2 coincide. There are infinitely many solutions.

Each of the points

$$(x, y) = (1, 1), (2, 3), (3, 5)$$

is a solution for this system.

The general solution: $x = t, y = -1 + 2t, t \in R$.

The solution set is

$$\{(x, y) \mid x = t, y = -1 + 2t, t \in R\} \text{ or } \{(t, -1 + 2t) \mid t \in R\}.$$

A system of linear equations has either

- 1) No solution, or
- 2) (Unique) Exactly one solution, or
- 3) Infinitely many solutions.

A system of linear equations is called **consistent** if it has at least one solution, and **inconsistent** if it has no solution.

Row Echelon Forms (REF)

A **leading entry** of a row refers to the left most non-zero entry (in a non-zero row).

Echelon form (or row echelon form):

1. Any rows consisting entirely of zeros are placed at the bottom of the matrix.
2. The first non-zero element in each row is positioned to the right of the first non-zero element in the previous row.

Example:

$$\begin{aligned} \text{i)} \quad & \begin{bmatrix} 2 & 1 & 6 & 4 & -2 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} & \text{ii)} \quad & \begin{bmatrix} 4 & 3 & -3 & 5 & 0 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 1 & 8 & -4 & 6 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} & \text{iii)} \quad & \begin{bmatrix} 2 & 5 & 3 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \text{iv)} \quad & \begin{bmatrix} 0 & 2 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 5 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & -1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 7 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & * \end{bmatrix} \end{aligned}$$

Some terms that we use

pivot position: a position of a leading entry in an echelon form of a matrix.

pivot: a non-zero number that is in a pivot position.

pivot column: a column that contains a pivot position.

basic variable: a variable that corresponds to a pivot column.

free variable: a variable that is not a basic variable, i.e, a variable that corresponds to a non-pivot column.

Note: In an echelon form, in each column that contains a leading entry of some row, all entries below the leading entry are zero.

Reduced Row Echelon Forms (RREF)

A matrix A is said to be in reduced row echelon form if it satisfies the following conditions.

- It is in row echelon form.
- The leading entry in each nonzero row is 1.
- Each leading 1 is the only nonzero entry in its column.

Examples:

$$\text{i)} \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \quad (\text{RREF})$$

$$\text{ii)} \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 1 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

(RREF if $a_{13} = 0$, not RREF if $a_{13} \neq 0$).

$$\text{iii)} \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & -1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \quad (\text{ not RREF})$$

Note: * can be any number.

$$\text{iv)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ (RREF)}$$

$$\text{v)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ (RREF)}$$

$$\text{vi)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ (not RREF, not REF)}$$

REF of a matrix is not unique:

Example:

$$\begin{bmatrix} 3 & 1 & 0 & -11 \\ 2 & 1 & -1 & -4 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 3 & 0 \end{bmatrix} \text{ (REF)}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ (REF)}$$

$$\sim \begin{bmatrix} 3 & 3 & 3 & 9 \\ 0 & 2 & 6 & 20 \\ 0 & 0 & 3 & 0 \end{bmatrix} \text{ (REF)}$$

RREF of a matrix is unique:

$$A = \begin{bmatrix} 3 & 1 & 0 & -11 \\ 2 & 1 & -1 & -4 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{(RREF)}} = B$$

The matrix B is the only **RREF** of A .

Elementary Row Operations:

i) Multiply a row by a non-zero scalar: $R'_2 = 5R_2$

ii) Interchange any two rows: $R_1 \longleftrightarrow R_3$

iii) Replace a row by the sum of itself and a multiple of another row: $R'_3 = R_3 + 4R_2$

Example: Solve the following system.

$$\begin{aligned}x + 2y &= 4 \\ -x + 3y + 3z &= -2 \\ y + z &= 0.\end{aligned}$$

Solution: The coefficient matrix of the system is

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

Its size is 3×3 (3 rows and 3 columns).

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array} \right].$$

Its size is 3×4 (3 rows and 4 columns).

We start with the augmented matrix.

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array} \right] R'_2 = R_2 + R_1 & \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 5 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{array} \right] R_2 \longleftrightarrow R_3 \\ & \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 3 & 2 \end{array} \right] R'_3 = R_3 - 5R_2 \\ & \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{array} \right] (REF). \end{aligned}$$

Since each column of the coefficient matrix has a pivot, all the variables are basic.

Thus, there is a unique solution.

Corresponding linear system is

$$\begin{aligned}x + 2y &= 4 \\ y + z &= 0 \\ -2z &= 2,\end{aligned}$$

which has the solution

$$\begin{aligned}z &= -1, \\ y &= -z = 1, \\ x &= -2y + 4 = -2 + 4 = 2.\end{aligned}$$

The solution is $(x, y, z) = (2, 1, -1)$.

The way that we solve the system here is called **back substitution**.

We can find the **RREF** of the augmented matrix and use it to find the solution as well.

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{array} \right] R'_3 = -\frac{1}{2}R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right] R'_2 = R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right] R'_1 = R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right] \quad (\text{RREF})$$

The solution is $(x, y, z) = (2, 1, -1)$.

Example: Determine whether the following linear system has a solution.

$$\begin{aligned} x + 2y + z &= 3 \\ x - y + z &= 1 \\ -2x - 4y - 2z &= 4 \end{aligned}$$

Solution: The augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ -2 & -4 & -2 & 4 \end{array} \right] \begin{array}{l} R'_2 = R_2 - R_1 \\ R'_3 = R_3 + 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 0 & 10 \end{array} \right]$$

The last row says

$$0 \cdot x + 0 \cdot y + 0 \cdot z = 10,$$

which is impossible.

The system does not have any solutions.

Example: Solve the following system of linear equations:

$$\begin{aligned}x + 2y - 3z &= 3 \\ -2x - 5y + 4z &= 5 \\ -5x - 13y + 9z &= 18\end{aligned}$$

Solution:

$$\begin{aligned}\left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ -2 & -5 & 4 & 5 \\ -5 & -13 & 9 & 18 \end{array} \right] \begin{array}{l} R'_2 = R_2 + 2R_1 \\ R'_3 = R_3 + 5R_1 \end{array} &\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & -3 & -6 & 33 \end{array} \right] \begin{array}{l} R'_3 = R_3 - 3R_2 \end{array} \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} R'_1 = R_1 + 2R_2 \end{array} \\ &\sim \left[\begin{array}{ccc|c} 1 & 0 & -7 & 25 \\ 0 & -1 & -2 & 11 \\ 0 & 0 & 0 & 0 \end{array} \right] (RREF).\end{aligned}$$

x and y are basic variables and z is free variable.

$$\begin{aligned}x - 7z = 25 &\implies x = 25 + 7z, \\ y + 2z = -11 &\implies y = -11 - 2z.\end{aligned}$$

The general solution in parametric vector form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25 + 7z \\ -11 - 2z \\ z \end{bmatrix} = \begin{bmatrix} 25 \\ -11 \\ 0 \end{bmatrix} + z \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix}, \quad z \in \mathbb{R}.$$

The system is consistent, and it has infinitely many solutions.

We can write the system

$$\begin{aligned} x + 2y - 3z &= 3 \\ -2x - 5y + 4z &= 5 \quad (*) \text{ (linear system)} \\ -5x - 13y + 9z &= 18 \end{aligned}$$

in the form of

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -5 & 4 \\ -5 & -13 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 18 \end{bmatrix} \quad (**) \text{ (matrix equation)}$$

or

$$x \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + y \begin{bmatrix} 2 \\ -5 \\ -13 \end{bmatrix} + z \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 18 \end{bmatrix} \quad (***) \text{ (vector equation)}.$$

Solving the linear system (*) is the same as solving the matrix equation (**), or vector equation (***)

Warning: Zero row in an augmented matrix always does not mean infinitely many solutions:

$$\bullet \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -3 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] : \text{ There are no solutions.}$$

$$\bullet \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] : \text{ There is a unique solution.}$$

$$\bullet \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 4 & 5 & 3 \\ 0 & 1 & 3 & 2 & -2 & 4 \\ 0 & 0 & 0 & 4 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] : \text{ There are infinitely many solutions.}$$

Example: Find the value of the constant k such that the following system has

- i) no solution,
- ii) infinitely many solutions,
- iii) unique solution.

$$\begin{aligned}x + 2y - z &= 1 \\ -2x - 3y + 2z &= -1 \\ -5x - 8y + 5z &= k\end{aligned}$$

Solution:

$$\begin{aligned}\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -2 & -3 & 2 & -1 \\ -5 & -8 & 5 & k \end{array} \right] \begin{array}{l} R'_2 = R_2 + 2R_1 \\ R'_3 = R_3 + 5R_1 \end{array} &\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 5+k \end{array} \right] \begin{array}{l} R'_1 = R_1 - 2R_2 \\ R'_3 = R_3 - 2R_2 \end{array} \\ &\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3+k \end{array} \right].\end{aligned}$$

- i) $k \neq -3$.
- ii) $k = -3$.
- iii) There is no k such that the system has a unique solution.

Example: Solve the **homogeneous** system:

$$\begin{aligned}x + 3y - 2z - w &= 0 \\ -2x - 5y + 4w &= 0 \\ x + 4y - 6z + w &= 0\end{aligned}$$

Solution:

$$\begin{aligned}\left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ -2 & -5 & 0 & 4 & 0 \\ 1 & 4 & -6 & 1 & 0 \end{array} \right] \begin{array}{l} R'_2 = R_2 + 2R_1 \\ R'_3 = R_3 - R_1 \end{array} &\sim \left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 1 & -4 & 2 & 0 \end{array} \right] \begin{array}{l} R'_3 = R_3 - R_2 \end{array} \\ &\sim \left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].\end{aligned}$$

$$y = 4z - 2w,$$

$$x = -3y + 2z + w = -3(4z - 2w) + 2z + w = -10z + 7w.$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -10z + 7w \\ 4z - 2w \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix}; \quad z, w \in R.$$

Example: Solve the **non-homogeneous** system:

$$x + 3y - 2z - w = 2$$

$$-2x - 5y + 4w = 3$$

$$x + 4y - 6z + w = 9$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 2 \\ -2 & -5 & 0 & 4 & 3 \\ 1 & 4 & -6 & 1 & 9 \end{array} \right] \begin{array}{l} R'_2 = R_2 + 2R_1 \\ R'_3 = R_3 - R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 2 \\ 0 & 1 & -4 & 2 & 7 \\ 0 & 1 & -4 & 2 & 7 \end{array} \right] \begin{array}{l} R'_3 = R_3 - R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 2 \\ 0 & 1 & -4 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

$$y = 4z - 2w + 7,$$

$$x = -3y + 2z + w + 2 = -3(4z - 2w + 7) + 2z + w + 2 = -10z + 7w - 19.$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -10z + 7w - 19 \\ 4z - 2w + 7 \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -19 \\ 7 \\ 0 \\ 0 \end{bmatrix}; \quad z, w \in \mathbb{R}.$$

$$\left\{ \begin{array}{l} \text{general solution} \\ \text{of the corresponding} \\ \text{homogeneous system} \end{array} \right\} = v_h = z \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix},$$

$$\left\{ \begin{array}{l} \text{a particular solution of the} \\ \text{non-homogeneous system} \end{array} \right\} = p = \begin{bmatrix} -19 \\ 7 \\ 0 \\ 0 \end{bmatrix}.$$

The general solution of a non-homogeneous system is $\boxed{X = v_h + p}$.

Example: Describe all solutions of the homogeneous equation $Ax = 0$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 6 & 0 & 8 & -1 & -2 \\ 0 & 0 & 1 & -3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: The augmented matrix is

$$\left[\begin{array}{cccccc|c} 1 & 6 & 0 & 8 & -1 & -2 & 0 \\ 0 & 0 & 1 & -3 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]. \quad \begin{array}{l} (x_1, x_3, x_6 \text{ are basic variables}) \\ (x_2, x_4, x_5 \text{ are free variables}) \end{array}$$

The general solution (in parametric vector form) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -6x_2 - 8x_4 + x_5 \\ x_2 \\ 3x_4 - 4x_5 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad x_2, x_4, x_5 \in R.$$

Definition: If $v_1, v_2, v_3, \dots, v_p$ are vectors in R^n , and $c_1, c_2, c_3, \dots, c_p$ are scalars, then the vector x defined by

$$x = c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_pv_p$$

is called a **linear combination** of $v_1, v_2, v_3, \dots, v_p$.

The set of all linear combinations of v_1, v_2, \dots, v_p is called the **span** of $v_1, v_2, v_3, \dots, v_p$ and denoted by $\text{Span}\{v_1, v_2, v_3, \dots, v_p\}$.

- Scalar multiples of v_i are in $\text{Span}\{v_1, v_2, \dots, v_p\}$.
- Zero vector is in $\text{Span}\{v_1, v_2, \dots, v_p\}$.
- $\text{Span}\{v_1\} = \{cv_1 | c \in R\}$.
- $\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is the x -axis.

Example: Determine if the vector $b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is a linear combination of the vectors

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}.$$

Solution:

$$b \in \text{Span}\{v_1, v_2, v_3\} \iff xv_1 + yv_2 + zv_3 = b$$

has a solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

System is consistent $\implies b \in \text{Span}\{v_1, v_2, v_3\}$.

The general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 - 5z \\ 3 - 4z \\ z \end{bmatrix}, z \in R,$$

which means that for any scalar z ,

$$(2 - 5z)v_1 + (3 - 4z)v_2 + zv_3 = b.$$

For $z = -1$: $7v_1 + 7v_2 - v_3 = b$.

For $z = 0$: $-2v_1 + 3v_2 + 0v_3 = b \implies -2v_1 + 3v_2 = b$.

For $z = 1$: $-3v_1 - v_2 + v_3 = b$.

For $z = 3$: $-8v_1 - 5v_2 + 2v_3 = b$.

Example: Let

$$b = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}.$$

For what value(s) of h , b is in $\text{Span}\{v_1, v_2\}$?

Solution:

$$\begin{aligned} \left[\begin{array}{cc|c} v_1 & v_2 & b \end{array} \right] &= \left[\begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{array} \right] R_3' = R_3 + 2R_1 \\ &\sim \left[\begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & 2h - 5 \end{array} \right] R_3' = R_3 - 3R_2 \\ &\sim \left[\begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 2h + 4 \end{array} \right]. \end{aligned}$$

System is consistent $\iff 2h + 4 = 0$.

So, if $h = -2$, then b is in $\text{Span}\{v_1, v_2\}$.

Note: If $h = -2$, then

$$\begin{bmatrix} -2 \\ -3 \\ -5 \end{bmatrix} = -8 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}.$$

Example: Solve the following system:

$$\begin{aligned} 2x + 4y - 6z &= 2 \\ y + 3z &= 5 \\ -3x - 5y + 7z &= -3 \end{aligned}$$

Solution:

$$\begin{aligned} \left[\begin{array}{ccc|c} 2 & 4 & -6 & 2 \\ 0 & 1 & 3 & 5 \\ -3 & -5 & 7 & -3 \end{array} \right] R_1' = \frac{1}{2}R_1 &\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 5 \\ -3 & -5 & 7 & -3 \end{array} \right] R_3' = R_3 + 3R_1 \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & -2 & 0 \end{array} \right] R_1' = R_1 - 2R_2 \\ &\quad R_3' = R_3 - R_2 \\ &\sim \left[\begin{array}{ccc|c} 1 & 0 & -9 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -5 & -5 \end{array} \right] R_3' = -\frac{1}{5}R_3 \\ &\sim \left[\begin{array}{ccc|c} 1 & 0 & -9 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array} \right] R_1' = R_1 + 9R_3 \\ &\quad R_2' = R_2 - 3R_3 \\ &\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right]. \end{aligned}$$

The solution of the system: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$

Example: Solve the matrix equations

$$AX = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \text{ where } A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -2 \\ -2 & 3 & 3 \end{bmatrix}.$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 3 & 7 & -2 & 1 \\ -2 & 3 & 3 & -1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -16 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -14 \end{array} \right].$$

The solution of the system: $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -16 \\ 3 \\ -14 \end{bmatrix}.$

Theorem: The following statements are equivalent for any $m \times n$ matrix A .

- For each b in R^m , the equation $Ax = b$ has a solution.
- Each $b \in R^m$ is a linear combination of the columns of A .
- The columns of A span R^m .
- A has a pivot position in every row.

The columns of A span $R^m \iff$ Any vector in R^m can be written as a linear combination of the columns of A .

Linear Dependence and Independence

Definition: A set of vectors $\{v_1, v_2, \dots, v_k\}$ in R^n is said to be linearly independent if the vector equation

$$c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$$

has only the trivial solution, i.e, $c_1 = c_2 = \dots = c_k = 0$.

The set $\{v_1, v_2, \dots, v_k\}$ in R^n is said to be linearly dependent if there exist scalars c_1, c_2, \dots, c_k , not all zero, such that $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$.

Example: Decide if the set of vectors $\left\{ \begin{bmatrix} 12 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

Solution: We consider the vector equation

$$c_1 \begin{bmatrix} 12 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Take $c_1 = 1$ and $c_2 = 3$. Then

$$\begin{bmatrix} 12 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, v_1 and v_2 are linearly dependent.

A set of two vectors $\{v_1, v_2\}$ is linearly dependent if and only if one is a scalar multiple of the other.

Example: Decide if the following vectors are linearly independent.

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

Solution: Consider the vector equation $xv_1 + yv_2 + zv_3 = 0$.

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 2z \\ z \end{bmatrix}, z \in R.$$

Choose $z = 1$. Then

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The vectors are linearly dependent.

Note: None of the vectors is a multiple of one of the other vectors.

Theorem: If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, v_2, \dots, v_p\}$ in R^n is linearly dependent if $p > n$.

Example: Decide if the following vectors are linearly independent.

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix}$$

Solution: Consider the equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$.

$$\left[\begin{array}{ccc|c} v_1 & v_2 & v_3 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 3 & -5 & 5 & 0 \\ -2 & 6 & -6 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & -6 & 0 \end{array} \right].$$

Since each column is a pivot column, there is a unique solution which is

$$x_1 = 0, x_2 = 0, x_3 = 0.$$

The given vectors are linearly independent.

Example: Decide if the following vectors are linearly independent.

$$v_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ -7 \\ 3 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 7 \\ -2 \\ -6 \end{bmatrix} \in R^4.$$

Solution: Consider the equation $c_1v_1 + c_2v_2 + c_3v_3 = 0$.

$$\left[\begin{array}{ccc|c} v_1 & v_2 & v_3 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 3 & 4 & 3 & 0 \\ -1 & -7 & 7 & 0 \\ 1 & 3 & -2 & 0 \\ 0 & 2 & -6 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Then, $c_1 = c_2 = c_3 = 0$. The vectors are linearly independent.

Summary: We put the given vectors in a matrix A as columns.

Then we find an REF of A , say B .

- If B has a pivot position in each column, then the vectors are linearly independent.
- If B has a non-pivot column, then the vectors are linearly dependent.

Examples:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 5 & 7 \\ 0 & 0 & 3 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Columns of A are linearly independent.
- Columns of A do not span R^3 .
- Columns of B are linearly dependent.
- Columns of B span R^3 .
- Columns of C are linearly independent.
- Columns of C span R^3 .
- Columns of D are linearly dependent.
- Columns of D do not span R^4 .

Example: Are the following vectors linearly independent?

$$\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ 10 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 3 & -2 & 0 \\ 3 & 10 & -7 & 1 \\ -5 & -5 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix}.$$

- Since there is a column which does not have a pivot position, the vectors are linearly dependent.
- Since the number of vectors is more than the number of entries in one vector, vectors are linearly dependent.

In R^3 , the maximum number of linearly independent vectors is 3.

Example: Decide if the following vectors are linearly dependent.

If yes, express one of the vectors as a linear combination of the other vectors.

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 1 & 0 & 4 \\ -1 & 0 & 3 & -1 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a non-pivot column, vectors are linearly dependent.

x_1 , x_2 and x_3 are basic variables, x_4 is free variable. The general solution of the equation $Ax = b$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4x_4 \\ 0 \\ -x_4 \\ x_4 \end{bmatrix}, \quad x_4 \in R.$$

If we choose $x_4 = 1$, then we have: $-4v_1 + 0v_2 - v_3 + v_4 = 0$.

$$\begin{aligned} v_4 &= 4v_1 + v_3 \\ v_1 &= -\frac{1}{4}v_3 + \frac{1}{4}v_4 \\ v_3 &= -4v_1 + v_4 \end{aligned}$$

The vector v_2 cannot be expressed as a linear combination of the vectors v_1 , v_3 and v_4 .

Remark: A set S with two or more vectors is linearly dependent \iff at least one of the vectors in S is expressible as a linear combination of other vectors in S .

Example: Find the value(s) of h for which the vectors

$$\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ h \end{bmatrix}$$

are linearly dependent.

Solution:

$$\begin{bmatrix} 1 & -2 & -1 \\ 3 & -4 & 1 \\ -3 & 1 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 7+h \end{bmatrix}.$$

For $h = -7$, the vectors are linearly dependent.

If $h \in R$ and $h \neq -7$, then the vectors are linearly independent.

Example: Find the value(s) of h for which the vectors

$$\begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ h \\ -8 \end{bmatrix}$$

are linearly dependent.

Solution:
$$\begin{bmatrix} 1 & -3 & 4 \\ -5 & 8 & h \\ -2 & 6 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 \\ 0 & -7 & 20+h \\ 0 & 0 & 0 \end{bmatrix}.$$

Vectors are linearly dependent for all h in R .

Remarks:

- A set S with two or more vectors is linearly independent \iff no vector in S is expressible as a linear combination of other vectors in S .
- Zero vector is linearly dependent.
Any set of vectors containing zero vector is linearly dependent.
- Any set $\{v_1, v_2, \dots, v_k\}$ in R^n is linearly dependent if $k > n$, i.e., $\#$ of vectors $>$ $\#$ of entries in one vector.
- Columns of a matrix A are linearly independent $\iff AX = 0$ has only the trivial solution.